On several partition problems of Bollobás and Scott

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Abstract

Judicious partition problems on graphs and hypergraphs ask for partitions that optimize several quantities simultaneously. Let G be a hypergraph with m_1 edges of size i for $i = 1, 2$. We show that for any integer $k \geq 1$, $V(G)$ admits a partition into k sets each containing at most $m_1/k + m_2/k^2 + o(m_2)$ edges, establishing a conjecture of Bollobás and Scott. We also prove that $V(G)$ admits a partition into $k \geq 3$ sets, each meeting at least $m_1/k + m_2/(k-1) + o(m_2)$ edges, which for large graphs implies a conjecture of Bollobás and Scott (the conjecture is for all graphs). For $k = 2$, we prove that $V(G)$ admits a partition into two sets each meeting at least $m_1/2 + 3m_2/4 + o(m_2)$ edges, which solves a special case of a more general problem of Bollobás and Scott.

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1 Introduction

Classical graph partition problems often ask for partitions of a graph that optimize a single quantity. For example, the well-known *Max-Cut Problem* asks for a partition V_1, V_2 of $V(G)$, where G is a weighted graph, that maximizes the total weight of edges with an end in each V_i. This problem is NP-hard, see [13]. The unweighted version is often called the *Maximum Bipartite Subgraph Problem*: Given a graph G find a partition V_1, V_2 of $V(G)$ that maximizes $e(V_1, V_2)$, the number of edges between V_1 and V_2. This is also NP-hard. However, it is easy to prove that any graph with m edges has a partition V_1, V_2 with $e(V_1, V_2) \geq m/2$. Edwards [10,11] improved this lower bound to $m/2 + \frac{1}{4}(\sqrt{2m + 1/4} - 1/2)$. This is best possible, as K_{2n+1} are extremal graphs.

In practice one often needs to find a partition of a given graph to optimize several quantities simultaneously. Such problems are called *Judicious Partition Problems* by Bollobás and Scott [4]. One such example is the problem of finding a partition V_1, V_2 of the vertex set of a graph G that minimizes $\max\{e(V_1), e(V_2)\}$, where $e(V_i)$ denotes the number of edges of G with both ends in V_i. This problem is also known as the *Bottleneck Bipartition Problem*, raised by Entringer (see, for example, [14,15]). Shahrokhi and Székely [16] showed that this problem is NP-hard. Porter [14] proved that any graph with m edges has a partition into V_1, V_2 with $e(V_i) \leq m/4 + O(\sqrt{m})$. Bollobás and Scott [6] improved this to $e(V_i) \leq m/4 + \frac{1}{8}(\sqrt{2m + 1/4} - 1/2)$, and showed that K_{2n+1} are the only extremal graphs.

In fact, Bollobás and Scott [6] proved that any graph with m edges has a partition V_1, V_2 such that $e(V_1, V_2) \geq m/2 + \frac{1}{4}(\sqrt{2m + 1/4} - 1/2)$ and for $i = 1, 2$, $e(V_i) \leq m/4 + \frac{1}{8}(\sqrt{2m + 1/4} - 1/2)$. Alon *et al.* [1] showed that there is a connection between the Maximum Bipartite Subgraph Problem and the Bottleneck Bipartition Problem. More precisely, they proved the following: Let G be a graph with m edges and largest cut of size $m/2 + \delta$. If $\delta \leq m/30$ then $V(G)$ admits a partition V_1, V_2 such that $e(V_i) \leq m/4 - \delta/2 + 10\delta^2/m + 3\sqrt{m}$; and if $\delta \geq m/30$ then $V(G)$ admits a partition V_1, V_2 such that $e(V_i) \leq m/4 - m/100$. It would be interesting to know whether this result can be generalized to k-partitions.

One of the early problems about judicious partitions is the conjecture of Bollobás and Thomason (see [3,5,7,8]) that if G is an r-uniform hypergraph with m edges then $V(G)$ has a partition into $V_1, \ldots, V_r$ such that $d(V_i) \geq rm/(2r-1)$ for $i = 1, \ldots, r$, where $d(V_i)$ denotes the number of edges of G meeting V_i (i.e., contains at least one vertex of V_i). A natural approach to this problem is to find a reasonable partition, and remove vertices of one set and try to partition the remaining vertices into $r-1$ parts in a better way. This approach is used in [7] by Bollobás and Scott to partition 3-uniform hypergraphs.

In this paper, we study several judicious partition problems about graphs with requirement on edges as well as on vertices, and such problems are called mixed partition problems. We follow Bollobás and Scott [8] to use the term "hypergraphs with edges of size at most 2".

We show in Section 2 that if G is a hypergraph with m_i edges of size i, $i = 1, 2$, then $V(G)$ admits a partition V_1, V_2 such that $d(V_i) \geq m_1/2 + 3m_2/4 + o(m_2)$ for $i = 1, 2$. This settles a problem of Bollobás and Scott [8] for large graphs, where they suggest the lower bound $(m_1 - 1)/2 + 2m_2/3$ as a starting point for a more general problem. Note that if we take a random partition V_1, V_2, then $\mathbb{E}(d(V_i)) = m_1/2 + 3m_2/4$.

In Section 3 we attempt to generalize the results in Section 2 to k-partitions. In particular, we prove that if $k \geq 3$ and G is a hypergraph with m_i edges of size i, $i = 1, 2$, then $V(G)$ admits a partition $V_1, \ldots, V_k$ such that $d(V_i) \geq m_1/k + m_2/(k-1) + o(m_2)$ for $i = 1, \ldots, k$. Again, if we take a random partition $V_1, \ldots, V_k$, then $\mathbb{E}(d(V_i)) = m_1/k + (2k-1)m_2/k^2$. Bollobás and

Scott [7] conjectured that every graph with m edges has a partition into k sets, each meeting at least $2m/(2k-1)$ edges. Our result implies this conjecture for large graphs.

In Section 4 we consider a generalization of the Bottleneck Bipartition Problem. We show that if $k \geq 1$ and G is a hypergraph with m_i edges of size i, $i = 1, 2$, then $V(G)$ admits a partition $V_1, \ldots, V_k$ such that $e(V_i) \leq m_1/k + m_2/k^2 + o(m_2)$ for $i = 1, \ldots, k$, establishing a conjecture of Bollobás and Scott [8]. Note that for a random partition $V_1, \ldots, V_k$, we have $\mathbb{E}(e(V_i)) = m_1/k + m_2/k^2$. Also when $m_1 = o(m_2)$ this follows from equation (2) in [8].

The approach we take is similar to that of Bollobás and Scott [5]. We first partition a set of large degree vertices, then establish a random process to partition the remaining vertices, and finally apply a concentration inequality to bound the deviations. The key is to pick the probabilities appropriately so that the expectation of the process will be in a range that we want. This will be achieved by extremal techniques.

Some notation is in order. Let G be a hypergraph and $S \subseteq V(G)$. We use $G[S]$ to denote the subgraph of G consisting of S and all edges of G contained in S. Let A, B be subsets of $V(G)$ or subgraphs of G, we use (A, B) denote the set of edges of G that are contained in $A \cup B$ and intersect both A and B. For a set $X \subseteq V(G)$ we use $d(X)$ to denote the number of edges of G meeting X, i.e., containing at least one member of X.

We will actually prove partition results for weighted graphs. Let G be a graph and let $w : V(G) \cup E(G) \to \mathbf{R}^+$, where $\mathbf{R}^+$ is the set of nongentive reals. For $X \subseteq V(G)$ we write

$$w_G(X) = \sum_{u_i \in X} w(u_i) + \sum_{\{e \in E(G):\ e \subseteq X\}} w(e)$$

and

$$\tau_G(X) = \sum_{u_i \in X} w(u_i) + \sum_{\{e \in E(G):\ e \cap X \neq \emptyset\}} w(e).$$

If G is understood, we use $\tau(X), w(X)$ instead of $\tau_G(X), w_G(X)$, respectively. We point out that if H is an induced subgraph of G, then for any $X \subseteq V(H)$, we have $w_H(X) = w_G(X)$. Also, note that when $w(e) = 1$ for all $e \in E(G)$ and $w(v) = 0$ for all $v \in V(G)$, we have $\tau(X) = d(X)$.

2 Bipartitions

In this section we consider the following problem of Bollobás and Scott [8]: Given a hypergraph G with m_i edges of size i, $1 \leq i \leq 2$, does there exist a partition of $V(G)$ into sets V_1 and V_2 such that $d(V_i) \geq \frac{m_1-1}{2} + \frac{2}{3}m_2$ for $i = 1, 2$. This problem was motivated by the Bollobás-Thomason conjecture on r-uniform hypergraphs. Bollobás and Scott [8] proved that if G is a hypergraph with m_i edges of size i, $i = 1, \ldots, k$, then $V(G)$ admits a partition V_1, V_2 such that for $i = 1, 2$,

$$d(V_i) \geq \frac{m_1 - 1}{3} + \frac{2m_2}{3} + \ldots + \frac{km_k}{k+1}.$$

They then used this to show that every 3-uniform hypergraph with m edges can be partitioned into 3-sets each of which meets at least $5m/9$ edges.

In [7], Bollobás and Scott suggest that the following might hold: Given a hypergraph G with m_i edges of size i, $1 \leq i \leq k$, there exists a partition of $V(G)$ into sets $V_1, \ldots, V_k$ such that for $i = 1, \ldots, k$,

$$d(V_i) \geq \frac{m_1 - 1}{2} + \frac{2m_2}{3} + \ldots + \frac{km_k}{k+1}.$$

In fact, they suggest in [8] that asymptotically the bound may be much larger:

$$d(V_i) \geq \frac{m_1}{2} + \frac{3}{4}m_2 + \ldots + \left(1 - \frac{1}{2^k}\right) m_k + o(m_1 + \ldots + m_k).$$

In this section we confirm this for $k = 2$ (see Theorem 2.4). Note that by taking a random partition $V_1, \ldots, V_k$, we have $\mathbb{E}(d(V_i)) = \frac{m_1}{2} + \frac{3}{4}m_2 + \ldots + \left(1 - \frac{1}{2^k}\right) m_k$.

As mentioned in the previous section, we need a concentration inequality, the Azuma-Heoffding inequality [2, 12], to bound deviations. We use the version given in [5].

Lemma 2.1 *Let $Z_1, \ldots, Z_n$ be independent random variables taking values in $\{1, \ldots, k\}$, let $Z := (Z_1, \ldots, Z_n)$, and let $f : \{1, \ldots, k\}^n \to \mathbf{N}$ such that $|f(Y) - f(Y')| \leq c_i$ for any $Y, Y' \in \{1, \ldots, k\}^n$ which differ only in the ith coordinate. Then for any $z > 0$,*

$$\mathbb{P}\left(f(Z) \geq E(f(Z)) + z\right) \leq \exp\left(\frac{-z^2}{2\sum_{i=1}^k c_i^2}\right),$$

$$\mathbb{P}\left(f(Z) \leq E(f(Z)) - z\right) \leq \exp\left(\frac{-z^2}{2\sum_{i=1}^k c_i^2}\right).$$

We also need a simple lemma to be used to choose probabilities in a random process.

Lemma 2.2 *Let $a, b, n \in \mathbf{R}^+$ with $a + b > 0$. Then there exists $p \in [0, 1]$ such that*

$$\min\{(n+b)p + a, (n+a)(1-p) + b\} \geq \frac{n}{2} + \frac{3}{4}(a+b).$$

Proof. Setting $(n + b)p + a = (n + a)(1 - p) + b$, we obtain

$$p = \frac{n+b}{2n+a+b};$$

and hence

$$(n+b)p + a = \frac{(n+b)^2}{2n+a+b} + a.$$

Clearly $p \in [0, 1]$. It is straightforward to show that

$$\frac{(n+b)^2}{2n+a+b} + a - \left(\frac{n}{2} + \frac{3}{4}(a+b)\right) = \frac{(a-b)^2}{4(2n+a+b)} \geq 0.$$

Hence, the assertion of the lemma holds. ∎

Remark. We may take $p = \dfrac{n+b}{2n+a+b}$ in Lemma 2.2.

We now prove the main result in this section. This is a partition result on weighted graphs. Recall the notation $\tau(X)$ defined in the previous section.

Theorem 2.3 *Let G be a graph with n vertices and m edges and let $w : V(G) \cup E(G) \to \mathbf{R}^+$ such that $w(e) > 0$ for all $e \in E(G)$. Let $\lambda = \max\{w(x) : x \in V(G) \cup E(G)\}$, $w_1 = \sum_{v \in V(G)} w(v)$, and $w_2 = \sum_{e \in E(G)} w(e)$. Then there is a partition $V(G) = X \cup Y$ such that*

$$\min\{\tau(X), \tau(Y)\} \geq \frac{1}{2}w_1 + \frac{3}{4}w_2 + \lambda \cdot O(m^{4/5}).$$

Proof. We may assume that G is connected, since otherwise we may simply consider the individual components. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ such that $d(v_1) \geq d(v_2) \geq \cdots \geq d(v_n)$.

First, we need to deal with an appropriate number of vertices so that the remaining vertices will have small degree (and hence will be useful when applying the Azuma-Hoeffding inequality in Lemma 2.1). Since G is connected, $n-1 \leq m < \frac{1}{2}n^2$. Fix $0 < \alpha < \frac{1}{2}$ (to be optimized later), and let $V_1 = \{v_1, \ldots, v_t\}$ such that $t = \lfloor m^\alpha \rfloor$. (Note that, since $\alpha < 1/2$ and $m < \frac{1}{2}n^2$, we have $t < n$.) Then $e(V_1) \leq \binom{t}{2} < \frac{1}{2}t^2 \leq \frac{1}{2}m^{2\alpha}$. Since $\sum_{i=1}^{t+1} d(v_i) \leq 2m$, $d(v_{t+1}) \leq \frac{2m}{t+1} \leq 2m^{1-\alpha}$. Let $V_2 = V(G) \backslash V_1$, and rename the vertices in V_2 as $\{u_1, u_2, ..., u_{n-t}\}$ such that $e(\{u_i\}, V_1 \cup \{u_1, ..., u_{i-1}\}) > 0$ for $i = 1, \ldots, n-t$; which can be done since we assume that G is connected.

We now partition the vertices of G. First, fix a random partition $V_1 = X_0 \cup Y_0$, and assign color 1 to all vertices in X_0 and color 2 to all vertices in Y_0. The vertices $u_i \in V_2$ are independently colored 1 with probability p_i, and 2 with probability $1 - p_i$. (The p_i's are constants to be determined recursively.) Let Z_i denote the indicator random variable of the event of coloring u_i. Hence $Z_i = j$, $j \in \{1, 2\}$, iff u_i is assigned the color j. When this process stops we obtain a bipartition of $V(G)$ into two sets X, Y, where X consists of all vertices with color 1 and Y consists of all vertices of color 2 (and hence $X_0 \subseteq X$ and $Y_0 \subseteq Y$).

We need additional notation to facilitate the choices of p_i ($1 \leq i \leq n-t$), the computations of expectations of $\tau(X)$ and $\tau(Y)$, and the estimations of concentration bounds. Let $G_i = G[V_1 \cup \{u_1, u_2, ..., u_i\}]$ for $i = 1, \ldots, n-t$, let $G_0 = G[V_1]$, and let the elements of $V(G_i) \cup E(G_i)$ inherit their weights from G. Let $x_0 = \tau(X_0)$ and $y_0 = \tau(Y_0)$, and define, for $i = 1, \ldots, n-t$,

$$
\begin{aligned}
X_i &= \{\text{vertices of } G_i \text{ with color } 1\}, \\
Y_i &= \{\text{vertices of } G_i \text{ with color } 2\}, \\
x_i &= \tau_{G_i}(X_i), \\
y_i &= \tau_{G_i}(Y_i), \\
\Delta x_i &= x_i - x_{i-1}, \\
\Delta y_i &= y_i - y_{i-1}, \\
a_i &= \sum_{e \in (u_i, X_{i-1})} w(e), \\
b_i &= \sum_{e \in (u_i, Y_{i-1})} w(e).
\end{aligned}
$$

Note that x_i and y_i are random variables determined by $(Z_1, Z_2, \ldots, Z_i)$; and a_i and b_i are random variables determined by $(Z_1, Z_2, \ldots, Z_{i-1})$. Thus,

$$
\begin{aligned}
\mathbb{E}(\Delta x_i | Z_1, \ldots, Z_{i-1}) &= p_i(w(u_i) + b_i) + a_i, \\
\mathbb{E}(\Delta y_i | Z_1, \ldots, Z_{i-1}) &= (1 - p_i)(w(u_i) + a_i) + b_i.
\end{aligned}
$$

Hence,

$$\begin{aligned}
\mathbb{E}(\Delta x_i) &= \mathbb{E}\left(\mathbb{E}(\Delta x_i | Z_1, \ldots, Z_{i-1})\right) \\
&= \sum_{(Z_1,\ldots,Z_{i-1})} \mathbb{P}(Z_1, \ldots, Z_{i-1}) \left(p_i(w(u_i) + b_i) + a_i\right) \\
&= p_i \left(w(u_i) + \sum_{(Z_1,\ldots,Z_{i-1})} \mathbb{P}(Z_1, \ldots, Z_{i-1}) b_i \right) + \sum_{(Z_1,\ldots,Z_{i-1})} \mathbb{P}(Z_1, \ldots, Z_{i-1}) a_i.
\end{aligned}$$

Similarly,

$$\mathbb{E}(\Delta y_i) = (1 - p_i) \left(w(u_i) + \sum_{(Z_1,\ldots,Z_{i-1})} \mathbb{P}(Z_1, \ldots, Z_{i-1}) a_i \right) + \sum_{(Z_1,\ldots,Z_{i-1})} \mathbb{P}(Z_1, \ldots, Z_{i-1}) b_i.$$

Let

$$\begin{aligned}
\alpha_i &= \sum_{(Z_1,\ldots,Z_{i-1})} \mathbb{P}(Z_1, \ldots, Z_{i-1}) a_i, \\
\beta_i &= \sum_{(Z_1,\ldots,Z_{i-1})} \mathbb{P}(Z_1, \ldots, Z_{i-1}) b_i.
\end{aligned}$$

Then

$$\begin{aligned}
\mathbb{E}(\Delta x_i) &= p_i(w(u_i) + \beta_i) + \alpha_i, \\
\mathbb{E}(\Delta y_i) &= (1 - p_i)(w(u_i) + \alpha_i) + \beta_i.
\end{aligned}$$

Note that α_i, β_i are determined by $p_1, \ldots, p_{i-1}$, since a_i and b_i are determined by $Z_1, \ldots, Z_{i-1}$. Also note that $e_i := a_i + b_i = \sum_{e \in (u_i,\ G_{i-1})} w(e)$ is the total weight of edges in $(u_i, V(G_{i-1}))$, which is independent of $Z_1, \ldots, Z_{i-1}$ and is the same in both G and G_i. Further, $e_i > 0$ by our choice of u_i and the assumption that $w(e) > 0$ for all $e \in E(G)$. Hence,

$$\begin{aligned}
\alpha_i + \beta_i &= \sum_{(Z_1,\ldots,Z_{i-1})} \mathbb{P}(Z_1, \ldots, Z_{i-1})(a_i + b_i) \\
&= \sum_{(Z_1,\ldots,Z_{i-1})} \mathbb{P}(Z_1, \ldots, Z_{i-1}) e_i \\
&= e_i \\
&> 0.
\end{aligned}$$

Let $p_i = \dfrac{w(u_i) + \beta_i}{2w(u_i) + \alpha_i + \beta_i}$. Note that p_i is recursively defined (by $p_1, \ldots, p_{i-1}$), since α_i and β_i are determined by $p_1, \ldots, p_{i-1}$. It follows from Lemma 2.2 that $p_i \in [0, 1]$ and

$$\min\{\mathbb{E}(\Delta x_i), \mathbb{E}(\Delta y_i)\} \geq \frac{1}{2} w(u_i) + \frac{3}{4}(\alpha_i + \beta_i) = \frac{1}{2} w(u_i) + \frac{3}{4} e_i.$$

We can now bound the expectations of x_{n-t} and y_{n-t}:

$$\mathbb{E}(x_{n-t}) = \mathbb{E}(x_0) + \sum_{i=1}^{n-t} \mathbb{E}(\Delta x_i) \geq \mathbb{E}(x_0) + \frac{1}{2}\sum_{i=1}^{n-t} w(u_i) + \frac{3}{4}\sum_{i=1}^{n-t} e_i,$$
$$\mathbb{E}(y_{n-t}) = \mathbb{E}(y_0) + \sum_{i=1}^{n-t} \mathbb{E}(\Delta y_i) \geq \mathbb{E}(y_0) + \frac{1}{2}\sum_{i=1}^{n-t} w(u_i) + \frac{3}{4}\sum_{i=1}^{n-t} e_i.$$

Let $X = X_{n-t}$ and $Y = Y_{n-t}$. Then $X \cup Y = V(G)$ and $X \cap Y = \emptyset$. Note that $\tau(X) = x_{n-t}$, $\tau(Y) = y_{n-t}$, $\tau(X_0) = x_0 = \mathbb{E}(x_0)$, and $\tau(Y_0) = y_0 = \mathbb{E}(y_0)$. Also note that $w_2 = \sum_{e\subseteq V_1} w(e) + \sum_{i=1}^{n-t} e_i$. Hence

$$\begin{aligned}
\mathbb{E}(\tau(X)) &\geq \frac{1}{2}\left(w_1 - \sum_{i=1}^{t} w(v_i)\right) + \frac{3}{4}\left(w_2 - \sum_{e\subseteq V_1} w(e)\right) + \tau(X_0)\\
&\geq \frac{1}{2}w_1 + \frac{3}{4}w_2 - \left(\frac{1}{2}\sum_{i=1}^{t} w(v_i) + \frac{3}{4}\sum_{e\subseteq V_1} w(e)\right)\\
&\geq \frac{1}{2}w_1 + \frac{3}{4}w_2 - \lambda\left(\frac{1}{2}t + \frac{3}{4}e(V_1)\right)\\
&\geq \frac{1}{2}w_1 + \frac{3}{4}w_2 - \lambda\left(\frac{1}{2}m^{\alpha} + \frac{3}{8}m^{2\alpha}\right).
\end{aligned}$$

Similarly,

$$\mathbb{E}(\tau(Y)) \geq \frac{1}{2}w_1 + \frac{3}{4}w_2 - \lambda\left(\frac{1}{2}m^{\alpha} + \frac{3}{8}m^{2\alpha}\right).$$

Next we show that $\tau(X)$ and $\tau(Y)$ are concentrated around their respective means. Note that changing the color of some u_i would affect $\tau(X)$ and $\tau(Y)$ by at most $d(u_i)\lambda + w(u_i) \leq (d(u_i)+1)\lambda$. Hence by applying Lemma 2.1, we have

$$\begin{aligned}
\mathbb{P}\left(\tau(X) < \mathbb{E}(\tau(X)) - z\right) &\leq \exp\left(-\frac{z^2}{2\lambda^2\sum_{i=1}^{n-t}(d(u_i)+1)^2}\right)\\
&\leq \exp\left(-\frac{z^2}{2\lambda^2\sum_{i=1}^{n-t}(d(u_i)+1)\cdot(d(v_{t+1})+1)}\right)\\
&< \exp\left(-\frac{z^2}{2\lambda^2(1+2m^{1-\alpha})\cdot(2m+n-1)}\right)\\
&< \exp\left(-\frac{z^2}{4\lambda^2 2m^{1-\alpha}\cdot(2m+m)}\right)\\
&= \exp\left(-\frac{z^2}{24\lambda^2 m^{2-\alpha}}\right).
\end{aligned}$$

Let $z = \lambda\sqrt{24\ln 2}\, m^{1-\frac{\alpha}{2}}$. Then

$$\mathbb{P}\left(\tau(X) < \mathbb{E}(\tau(X)) - z\right) < \frac{1}{2}$$

and

$$\mathbb{P}\left(\tau(Y) < \mathbb{E}(\tau(Y)) - z\right) < \frac{1}{2}.$$

So there exists a partition $V(G) = X \cup Y$ such that

$$\tau(X) \geq \mathbb{E}(\tau(X)) - z \geq \frac{1}{2}w_1 + \frac{3}{4}w_2 + \lambda \cdot o(m)$$

and

$$\tau(Y) \geq \mathbb{E}(\tau(Y)) - z \geq \frac{1}{2}w_1 + \frac{3}{4}w_2 + \lambda \cdot o(m).$$

The $o(m)$ term in the above expressions is

$$-\left(\frac{1}{2}m^{\alpha} + \frac{3}{8}m^{2\alpha} + \sqrt{24\ln 2}m^{1-\frac{\alpha}{2}}\right).$$

So picking $\alpha = 2/5$ to minimize $\max\{2\alpha, 1 - \frac{\alpha}{2}\}$, we have

$$\max\{\tau(X), \tau(Y)\} \geq \frac{1}{2}w_1 + \frac{3}{4}w_2 + \lambda \cdot O(m^{4/5}).$$

∎

When G is a hypergraph with edges of size 1 or 2, we may view G as a weighted graph with weight function w such that $w(e) = 1$ for all $e \in E(G)$ with $|e| = 2$, $w(v) = 1$ for all $v \in V(G)$ with $\{v\} \in E(G)$, and $w(v) = 0$ for all $v \in V(G)$ with $\{v\} \notin E(G)$. Theorem 2.3 then gives the following result.

Theorem 2.4 *Let G be a hypergraph with m_i edges of size i, $i = 1, 2$. Then there is a partition V_1, V_2 of $V(G)$ such that for $i = 1, 2$,*

$$d(V_i) \geq \frac{1}{2}m_1 + \frac{3}{4}m_2 + O(m_2^{4/5}).$$

As mentioned before a random bipartition shows that the expected value of $d(V_i)$ is $m_1/2 + 3m_2/4$.

3 k-Partitions – bounding edges meeting each set

In [7], Bollobás and Scott conjecture that every graph with m edges has a partition into k sets each of which meets at least $2m/(2k-1)$ edges. Note that in any k-partition of K_{2k-1}, one set consists of just one vertex, which meets $2m/(2k-1)$ edges; so the conjectured bound is best possible. For large graphs, it is likely that the bound is much better: a random k-partition $V_1, \ldots, V_k$ of a graph with m edges shows that $\mathbb{E}(d(V_i)) = (2k-1)m/k^2$.

For $k = 2$, the above conjecture is the $r = 2$ case of the Bollobás-Thomason conjecture on r-uniform hypergraphs; and it follows from the fact that every graph with m edges has a bipartition V_1, V_2 such that for $i \in \{1, 2\}$, each vertex in V_i has at least as many neighbors in V_{3-i} as in V_i. In this section, we prove this Bollobás-Scott conjecture for graphs when m is sufficiently large.

We use a similar approach as in the previous section, namely: First, partition an appropriate set of vertices of lager degree, then establish a martingale process to bound expectations, and finally apply the Azuma-Hoeffding inequality to bound deviations. As before, we need to pick probabilities for that process. To this end we need several lemmas. Our first lemma will be used to take care of critical points when applying the method of Lagrange multipliers to optimize a function.

Lemma 3.1 *Let $a_i = a > 0$ for $i = 1, \ldots, l$, and let $a_j = 0$ for $j = l+1, \ldots, k$, where $k \geq l \geq 2$. Let $\delta \geq 0$ and $\alpha_i = \left(\sum_{j=1}^k a_j\right) + \delta - a_i$. Then*

$$1 + \sum_{i=1}^k \frac{a_i}{\alpha_i} \geq \left(\frac{\delta}{k} + \frac{2k-1}{k^2} \sum_{i=1}^k a_i\right) \sum_{i=1}^k \frac{1}{\alpha_i}.$$

Proof. By the assumptions of the lemma, we have $\alpha_i = (l-1)a + \delta > 0$ for $1 \leq i \leq l$, and $\alpha_i = la + \delta > 0$ for $l+1 \leq i \leq k$. Let

$$f := 1 + \sum_{i=1}^k \frac{a_i}{\alpha_i} - \left(\frac{\delta}{k} + \frac{2k-1}{k^2} \sum_{i=1}^k a_i\right) \sum_{i=1}^k \frac{1}{\alpha_i}.$$

We need to prove $f \geq 0$. For convenience, let $\delta = a\varepsilon$. Then $\varepsilon \geq 0$ and

$$f = 1 + \frac{l}{l-1+\varepsilon} - \left(\frac{\varepsilon}{k} + \frac{2k-1}{k^2} l\right) \left(\frac{l}{l-1+\varepsilon} + \frac{k-l}{l+\varepsilon}\right).$$

A straightforward calculation shows that

$$(l-1+\varepsilon)(l+\varepsilon)f = \frac{l}{k^2}(k-1)(k-l) \geq 0.$$

Hence the assertion of the lemmas holds. ∎

Note that in the lemma below we are unable to gurantee $p_i \geq 0$ for all $i = 1, \ldots, k$; and hence these p_i cannot serve as probabilities in a random process. However, this lemma is needed in order to prove the next lemma.

Lemma 3.2 *Let $\delta \geq 0$ and, for $i = 1, \ldots, k$, let $a_i \geq 0$ and $\alpha_i = \left(\sum_{j=1}^k a_j\right) + \delta - a_i$. Then there exist $p_i, i = 1, \ldots, k$, such that $\sum_{i=1}^k p_i = 1$ and, for $1 \leq i \leq k$,*

$$\alpha_i p_i + a_i \geq \frac{\delta}{k} + \frac{2k-1}{k^2} \sum_{i=1}^k a_i.$$

Proof. For convenience let $f_i(p_1, \ldots, p_k) := \alpha_i p_i + a_i$, $i = 1, \ldots, k$. If $a_i = 0$ for $i = 1, \ldots, k$, then the assertion of the lemma holds by picking $p_i = 1/k$ for $i = 1, \ldots, k$. So without loss of generality we may assume $a_1 > 0$.

Now assume $a_i = 0$ for $i = 2, \ldots, k$. Then $f_1 = \delta p_1 + a_1$ and $f_i = (a_1 + \delta)p_i$ for $2 \leq i \leq k$. Setting $f_i = f_1$ for $i = 2, \ldots, k$, we get $p_i = \dfrac{\delta p_1 + a_1}{a_1 + \delta}$. Setting $\sum_{i=1}^k p_i = 1$, we have $p_1 = \dfrac{(2-k)a_1 + \delta}{a_1 + k\delta}$. Hence for $i = 1, \ldots, k$,

$$f_i = \delta p_1 + a_1 = \frac{(\delta + a_1)^2}{a_1 + k\delta},$$

and so,

$$f_i - \left(\frac{\delta}{k} + \frac{2k-1}{k^2}\sum_{i=1}^{k} a_i\right) = \frac{(k-1)^2 a_1^2}{(a_1+k\delta)k^2} \geq 0.$$

Therefore, we may further assume that $a_2 > 0$. Hence $\alpha_i > 0$ for all $i = 1, \ldots, k$. Setting $f_i = f_1$ for $i = 2, \ldots, k$, we get $p_i = \dfrac{\alpha_1 p_1 + a_1 - a_i}{\alpha_i}$ for $i = 1, \ldots, k$. Requiring $\sum_{i=1}^{k} p_i = 1$ and noting that $a_i - a_1 = \alpha_1 - \alpha_i$ for $1 \leq i \leq k$, we have

$$p_1 = \frac{1 + \sum_{i=1}^{k} \frac{a_i - a_1}{\alpha_i}}{\alpha_1 \sum_{i=1}^{k} \frac{1}{\alpha_i}} = \frac{1 + \sum_{i=1}^{k} \frac{\alpha_1 - \alpha_i}{\alpha_i}}{\alpha_1 \sum_{i=1}^{k} \frac{1}{\alpha_i}} = 1 - \frac{k-1}{\alpha_1 \sum_{i=1}^{k} \frac{1}{\alpha_i}}.$$

Indeed, for $j = 1, \ldots, k$,

$$p_j = 1 - \frac{k-1}{\alpha_j \sum_{i=1}^{k} \frac{1}{\alpha_i}}.$$

Note that $\alpha_j + a_j = \alpha_i + a_i$ for any $1 \leq i, j \leq k$. Hence for $j = 1, 2, \ldots, k$, we have

$$\begin{aligned} f_j &= \alpha_j p_j + a_j \\ &= \frac{\sum_{i=1}^{k} \frac{\alpha_j + a_j}{\alpha_i} - (k-1)}{\sum_{i=1}^{k} \frac{1}{\alpha_i}} \\ &= \frac{\sum_{i=1}^{k} \frac{\alpha_i + a_i}{\alpha_i} - (k-1)}{\sum_{i=1}^{k} \frac{1}{\alpha_i}} \\ &= \frac{1 + \sum_{i=1}^{k} \frac{a_i}{\alpha_i}}{\sum_{i=1}^{k} \frac{1}{\alpha_i}}. \end{aligned}$$

Now define

$$f(a_1, a_2, \ldots, a_k) := 1 + \sum_{i=1}^{k} \frac{a_i}{\alpha_i} - \left(\frac{\delta}{k} + \frac{2k-1}{k^2}\sum_{i=1}^{k} a_i\right)\sum_{i=1}^{k} \frac{1}{\alpha_i}.$$

To complete the proof of this lemma, we need to show $f(a_1, \ldots, a_k) \geq 0$.

Case 1. $\delta = 0$.

Then $\alpha_i + a_i = \sum_{j=1}^k a_j$ for $i = 1, \ldots, k$. Set $\alpha = \sum_{j=1}^k a_j$; then $\sum_{i=1}^k \alpha_i = (k-1)\alpha$. Moreover,

$$\begin{aligned} f(a_1, \ldots, a_k) &= 1 + \sum_{i=1}^k \frac{a_i}{\alpha_i} - \frac{(2k-1)\alpha}{k^2} \sum_{i=1}^k \frac{1}{\alpha_i} \\ &= 1 + \sum_{i=1}^k \frac{\alpha - \alpha_i}{\alpha_i} - \frac{(2k-1)\alpha}{k^2} \sum_{i=1}^k \frac{1}{\alpha_i} \\ &= \frac{(k-1)^2\alpha}{k^2} \sum_{i=1}^k \frac{1}{\alpha_i} - (k-1) \\ &\geq \frac{(k-1)^2\alpha}{k^2} \frac{k^2}{\sum_{i=1}^k \alpha_i} - (k-1) \\ &= 0. \end{aligned}$$

Here the inequality follws from Cauchy-Schwarz, and the last equality follows from the fact that $\sum_{i=1}^k \alpha_i = (k-1)\alpha$.

Case 2. $\delta > 0$.

Then $\alpha_i > 0$ for $i = 1, \ldots, k$. (So in this case we need not require $a_1 > 0$ and $a_2 > 0$.) Set $\alpha = \sum_{j=1}^k a_j$.

Let $g_l(a_1, \ldots, a_l) = f(a_1, \ldots, a_l, 0, \ldots, 0)$. It then suffices to show that $g_l(a_1, \ldots, a_l) \geq 0$ on the domain $D_l := [0, \alpha]^l$ for $l = 1, \ldots, k$.

First, we prove that for $l \in \{1, \ldots, k\}$, $g_l \geq 0$ at all possible critical points of g_l in D_l, subject to $\sum_{j=1}^k a_j - \alpha = 0$. For $j = 1, \ldots, l$,

$$\frac{\partial g_l}{\partial a_j} = -\sum_{i=1}^k \frac{a_i}{\alpha_i^2} + \frac{a_j}{\alpha_j^2} + \frac{1}{\alpha_j} + \frac{\delta}{k}\left(\sum_{i=1}^k \frac{1}{\alpha_i^2} - \frac{1}{\alpha_j^2}\right) - \frac{2k-1}{k^2}\left(\sum_{i=1}^k \frac{1}{\alpha_i} - \sum_{i=1}^k a_i \sum_{i=1}^k \frac{1}{\alpha_i^2} + \sum_{i=1}^k \frac{a_i}{\alpha_j^2}\right).$$

Using the method of Lagrange multipliers, we have $\frac{\partial g_l}{\partial a_j} = \lambda$ for all $j = 1, \ldots, l$. So $\frac{\partial g_l}{\partial a_j} = \frac{\partial g_l}{\partial a_1}$, which gives

$$\frac{a_j}{\alpha_j^2} + \frac{1}{\alpha_j} - \frac{\delta}{k}\frac{1}{\alpha_j^2} - \frac{2k-1}{k^2}\sum_{i=1}^k \frac{a_i}{\alpha_j^2} = \frac{a_1}{\alpha_1^2} + \frac{1}{\alpha_1} - \frac{\delta}{k}\frac{1}{\alpha_1^2} - \frac{2k-1}{k^2}\sum_{i=1}^k \frac{a_i}{\alpha_1^2}.$$

Since $\alpha_j + a_j = \alpha_1 + a_1 = \sum_{i=1}^k a_i + \delta$, we have

$$\frac{1}{\alpha_j^2}\left(\frac{(k-1)^2}{k^2}\sum_{i=1}^n a_i + \frac{k-1}{k}\delta\right) = \frac{1}{\alpha_1^2}\left(\frac{(k-1)^2}{k^2}\sum_{i=1}^n a_i + \frac{k-1}{k}\delta\right).$$

Hence $1/\alpha_j^2 = 1/\alpha_1^2$ for all $j = 1, \ldots, l$. Therefore, $\alpha_j = \alpha_1$ for $j = 1, \ldots, l$, which implies $a_j = a_1$ for $j = 1, \ldots, l$. It follows from Lemma 3.1 that $g_l \geq 0$ at all possible critical points of g_l in $[0, \alpha]^l$.

We now show that $g_l \geq 0$ on $[0, \alpha]^l$ by applying induction on l. Suppose $l = 1$. Then $\alpha = a_1$. So $\alpha_1 = \delta$, and $\alpha_i = a_1 + \delta$ for $i = 2, \ldots, k$. Hence

$$g_1(a_1) = 1 + \frac{a_1}{\delta} - \left(\frac{\delta}{k} + \frac{(2k-1)a_1}{k^2}\right)\left(\frac{1}{\delta} + \frac{k-1}{a_1 + \delta}\right) = \frac{(k-1)^2}{k^2}\left(\frac{a_1^2}{\delta(a_1 + \delta)}\right) \geq 0.$$

So we may assume $l \geq 2$ and $g_i \geq 0$ for all $i = 1, \dots, l-1$. We now show $g_l \geq 0$ on the domain $[0, \alpha]^l$ by proving it for all points in the boundary of $[0, \alpha]^l$ (since $g_l \geq 0$ at all possible critical points of g_l). Let $(a_1, \dots, a_l)$ be in the boundary of $[0, \alpha]^l$. Then $a_j = 0$ or $a_j = \alpha$ for some $j \in \{1, \dots, l\}$. Note that g_l is a symmetric function. So we may assume without loss of generality that $a_l = 0$ or $a_1 = \alpha$. If $a_l = 0$ then $g_l(a_1, \dots, a_l) = g_{l-1}(a_1, \dots, a_{l-1}) \geq 0$ by induction hypothesis. If $a_1 = \alpha$ then $a_j = 0$ for $j = 2, \dots, l$, and so, $g_l(a_1, \dots, a_l) = g_1(a_1) \geq 0$. Again, we have $g_l(a_1, \dots, a_l) \geq 0$. ∎

Note that, in the proof of Lemma 3.2, when $\alpha_i > 0$ for all $1 \leq i \leq k$ we have

$$p_j = 1 - \frac{k-1}{\alpha_j \sum_{i=1}^{k} \frac{1}{\alpha_i}}$$

for $j = 1, \dots, k$, which may be negative. We now apply Lemma 3.2 to prove the next result which gives the p_i's needed in a random process.

Lemma 3.3 *Let $\delta \geq 0$. For $i = 1, \dots, k$, where $k \geq 3$, let $a_i \geq 0$ and $\alpha_i = (\sum_{j=1}^{k} a_j) + \delta - a_i$. Then there exist $p_i \in [0, 1]$, $1 \leq i \leq k$, such that $\sum_{i=1}^{k} p_i = 1$ and for $1 \leq i \leq k$,*

$$\alpha_i p_i + a_i \geq \frac{\delta}{k} + \frac{1}{k-1} \sum_{i=1}^{k} a_i.$$

Proof. If $a_i = 0$ for $1 \leq i \leq k$, then $\alpha_i = \delta$ for $1 \leq i \leq k$, and it is easy to check that the assertion of the lemma holds by taking $p_i = 1/k$, $i = 1, \dots, k$. So we may assume without loss of generality that $a_1 > 0$. If $a_i = 0$ for $2 \leq i \leq k$ and $\delta = 0$, then $\alpha_1 = 0$ and $\alpha_i = a_1$ for $2 \leq i \leq k$; and the assertion of the lemma holds by setting $p_1 = 0$ and $p_i = \frac{1}{k-1}$ for $i = 2, \dots, k$. Therefore, we may further assume that $a_2 > 0$ or $\delta > 0$. As a consequence, we have $\alpha_i > 0$ for $1 \leq i \leq k$.

We prove the assertion of this lemma by induction on k. For $1 \leq i \leq k$, let

$$f_i(p_1, \dots, p_k) := \alpha_i p_i + a_i.$$

For $k = 3$, it follows from Lemma 3.2 (and the remark following its proof) that there exist p'_1, p'_2, p'_3 such that $p'_1 + p'_2 + p'_3 = 1$ and for $i = 1, 2, 3$,

$$p'_i = 1 - \frac{2}{\alpha_i \sum_{i=1}^{3} \frac{1}{\alpha_j}} \quad \text{and} \quad f_i(p'_1, p'_2, p'_3) \geq \frac{\delta}{3} + \frac{5}{9} \sum_{i=1}^{3} a_i.$$

If $p'_i \geq 0$ for $i = 1, 2, 3$, then the assertion of the lemma holds by taking $p_i := p'_i$, $i = 1, 2, 3$. So we may assume without loss of generality that $p'_3 < 0$, which implies $a_3 > \alpha_3 p'_3 + a_3 = f_3(p'_1, p'_2, p'_3) \geq \frac{\delta}{3} + \frac{5}{9} \sum_{i=1}^{3} a_i$. By Lemma 2.2 (with $n := a_3 + \delta$), there exist $p_1, p_2 \in [0, 1]$ such that $p_1 + p_2 = 1$ and

$$f_1(p_1, p_2, 0) = (a_2 + a_3 + \delta)p_1 + a_1 \geq \frac{a_3 + \delta}{2} + \frac{3}{4}(a_1 + a_2),$$
$$f_2(p_1, p_2, 0) = (a_1 + a_3 + \delta)p_2 + a_2 \geq \frac{a_3 + \delta}{2} + \frac{3}{4}(a_1 + a_2).$$

Now, let $p_3 = 0$. Then $p_1 + p_2 + p_3 = 1$, $p_i \in [0,1]$ for all $1 \le i \le 3$, and

$$f_1(p_1,p_2,p_3) = \alpha_1 p_1 + a_1 \ge \frac{\delta}{3} + \frac{1}{2}(a_1 + a_2 + a_3),$$
$$f_2(p_1,p_2,p_3) = \alpha_2 p_2 + a_2 \ge \frac{\delta}{3} + \frac{1}{2}(a_1 + a_2 + a_3),$$
$$f_3(p_1,p_2,p_3) = a_3 \ge \frac{\delta}{3} + \frac{1}{2}(a_1 + a_2 + a_3).$$

Hence Lemma 3.3 holds for $k = 3$.

Now let $n \ge 3$ be an integer, and assume that the assertion of the lemma holds when $k = n$. We prove the assertion of the lemma also holds when $k = n+1$. By Lemma 3.2 (and the remark following its proof), there exist p'_i, $1 \le i \le n+1$, such that $\sum_{i=1}^{n+1} p'_i = 1$ and for $i = 1, \dots, n+1$,

$$p'_i = 1 - \frac{n}{\alpha_i \sum\limits_{j=1}^{n+1} \frac{1}{\alpha_j}} \le 1,$$

and

$$f_i(p'_1, \dots, p'_{n+1}) \ge \frac{\delta}{n+1} + \frac{2n+1}{(n+1)^2} \sum_{i=1}^{n+1} a_i.$$

If $p'_i \ge 0$ for $1 \le i \le n+1$, then let $p_i := p'_i$; and the lemma holds (since $\frac{2n+1}{(n+1)^2} > \frac{1}{n}$ when $n \ge 3$). So we may assume without loss of generality that $p'_{n+1} < 0$. Then

$$\begin{aligned} a_{n+1} &> \alpha_{n+1} p'_{n+1} + a_{n+1} \\ &= f_{n+1}(p'_1, \dots, p'_{n+1}) \\ &\ge \frac{\delta}{n+1} + \frac{2n+1}{(n+1)^2} \sum_{i=1}^{n+1} a_i \\ &\ge \frac{\delta}{n+1} + \frac{1}{n} \sum_{i=1}^{n+1} a_i \end{aligned}$$

Let $\delta' = \delta + a_{n+1}$. Then for $1 \le i \le n$ we have $\alpha_i = (\sum_{j=1}^{n} a_j) + \delta' - a_i$. Hence by the induction hypothesis, there exist $p_i \in [0,1]$, $1 \le i \le n$, such that $\sum\limits_{i=1}^{n} p_i = 1$ and, for $i = 1, \dots, n$,

$$\begin{aligned} \alpha_i p_i + a_i &\ge \frac{\delta'}{n} + \frac{1}{n-1} \sum_{i=1}^{n} a_i \\ &= \frac{\delta}{n} + \frac{a_{n+1}}{n} + \frac{1}{n-1} \sum_{i=1}^{n} a_i \\ &\ge \frac{\delta}{n+1} + \frac{1}{n} \sum_{i=1}^{n+1} a_i. \end{aligned}$$

Let $p_{n+1} = 0$. Then $\sum\limits_{i=1}^{n+1} p_i = 1$ and $p_i \in [0,1]$ for all $1 \le i \le n+1$. Also,

$$f_i(p_1, \dots, p_{n+1}) \ge \frac{\delta}{n+1} + \frac{1}{n}\sum_{i=1}^{n+1} a_i, \text{ for } 1 \le i \le n,$$

$$f_{n+1}(p_1, \dots, p_{n+1}) = a_{n+1} \ge \frac{\delta}{n+1} + \frac{1}{n}\sum_{i=1}^{n+1} a_i.$$

Hence, Lemma 3.3 holds for $k = n+1$, completing the proof of this lemma. ∎

We can now prove the following partition result on weighted graphs.

Theorem 3.4 *Let $k \ge 3$ be an integer, let G be a graph with m edges, and let $w : V(G) \cup E(G) \to \mathbf{R}^+$ such that $w(e) > 0$ for all $e \in E(G)$. Let $\lambda = \max\{w(x) : x \in V(G) \cup E(G)\}$, $w_1 = \sum_{v \in V(G)} w(v)$ and $w_2 = \sum_{e \in E(G)} w(e)$. Then there is a partition $U_1, \dots, U_k$ of $V(G)$ such that for $1 \le i \le k$,*

$$\tau(U_i) \ge \frac{1}{k}w_1 + \frac{1}{k-1}w_2 + \lambda \cdot O(m^{4/5}).$$

Proof. We may assume that G is connected. We use the same notation as in the proof of Theorem 2.3. Let $V(G) = \{v_1, \dots, v_n\}$ such that $d(v_1) \ge d(v_2) \ge \dots \ge d(v_n)$. Let $V_1 = \{v_1, \dots, v_t\}$ with $t = \lfloor m^\alpha \rfloor$, where $0 < \alpha < 1/2$; and let $V_2 := V(G) \setminus V_1 = \{u_1, \dots, u_{n-t}\}$ such that $e(u_i, V_1 \cup \{u_1, \dots, u_{i-1}\}) > 0$ for $i = 1, \dots, n-t$. Then $e(V_1) \le \frac{1}{2}m^{2\alpha}$ and $d(v_{t+1}) \le 2m^{1-\alpha}$.

Fix a random partition $V_1 = Y_1 \cup Y_2 \cup \dots \cup Y_k$ and, for each $i \in \{1, \dots, k\}$, assign the color i to all vertices in Y_i. We extend this coloring to $V(G)$ such that each vertex $u_i \in V_2$ is independently assigned the color j with probability p^i_j, where $\sum_{j=1}^k p^i_j = 1$. Let Z_i be the indicator random variable of the event of coloring u_i, i.e., $Z_i = j$ iff u_i is colored j. Let $G_i = G[V_1 \cup \{u_1, \cdots, u_i\}]$ for $i = 1, \dots, n-t$, and let $G_0 = G[V_1]$. Let $X^0_j = Y_j$ and $x^0_j = \tau(X^0_j)$, and for $i = 1, \dots, n-t$ and $j = 1, \dots, k$, define

$$\begin{aligned} X^i_j &= \{\text{vertices of } G_i \text{ with color } j\}, \\ x^i_j &= \tau_{G_i}(X^i_j), \\ \Delta x^i_j &= x^i_j - x^{i-1}_j, \\ a^i_j &= \sum_{e \in (u_i, X^{i-1}_j)} w(e). \end{aligned}$$

Note that a^i_l is a random variable determined by $(Z_1, \dots, Z_{i-1})$. Hence, for $1 \le i \le n-t$ and $1 \le j \le k$,

$$\mathbb{E}(\Delta x^i_j | Z_1, \dots, Z_{i-1}) = p^i_j \left(\sum_{l=1}^k a^i_l + w(u_i) - a^i_j \right) + a^i_j.$$

So

$$\mathbb{E}(\Delta x^i_j) = p^i_j \left(\sum_{l=1}^k b^i_l + w(u_i) - b^i_j \right) + b^i_j,$$

where for $1 \leq l \leq k$,

$$b_l^i = \sum_{(Z_1,\ldots,Z_{i-1})} \mathbb{P}(Z_1, \ldots, Z_{i-1}) a_l^i.$$

Since a_l^i is determined by $(Z_1, \ldots, Z_{i-1})$, b_l^i is determined by p_j^s, $1 \leq s \leq i-1$ and $1 \leq j \leq k$.

By Lemma 3.3 (with $\delta = w(u_i)$), there exist $p_j^i \in [0,1]$, $1 \leq j \leq k$, such that $\sum_{j=1}^k p_j^i = 1$ and

$$\mathbb{E}(\Delta x_j^i) \geq \frac{w(u_i)}{k} + \frac{1}{k-1} \sum_{j=1}^k b_j^i.$$

Clearly, each p_j^i is dependent only on b_l^i, $1 \leq l \leq k$, and hence is determined (recursively) by p_l^s, $1 \leq l \leq k$ and $1 \leq s \leq i-1$. Note that $e_i := \sum\limits_{j=1}^k a_j^i = \sum_{e \in (u_i, G_{i-1})} w(e)$ is the total weight of the edges in (u_i, G_{i-1}), which is independent of $Z_1, \ldots, Z_{n-t}$. Thus,

$$\begin{aligned}
\mathbb{E}(\Delta x_j^i) &\geq \frac{w(u_i)}{k} + \frac{1}{k-1} \sum_{j=1}^k \sum_{(Z_1,\ldots,Z_{i-1})} \mathbb{P}(Z_1, \ldots, Z_{i-1}) a_j^i \\
&= \frac{w(u_i)}{k} + \frac{1}{k-1} \sum_{(Z_1,\ldots,Z_{i-1})} \left(\mathbb{P}(Z_1, \ldots, Z_{i-1}) \sum_{j=1}^k a_j^i \right) \\
&= \frac{w(u_i)}{k} + \frac{1}{k-1} \sum_{(Z_1,\ldots,Z_{i-1})} \mathbb{P}(Z_1, \ldots, Z_{i-1}) e_i \\
&= \frac{w(u_i)}{k} + \frac{1}{k-1} e_i.
\end{aligned}$$

Therfore, noting that $w_2 = \sum_{e \subseteq V_1} w(e) + \sum_{i=1}^{n-t} e_i$, we have

$$\begin{aligned}
\mathbb{E}(x_j^{n-t}) &= \sum_{i=1}^{n-t} \mathbb{E}(\Delta x_j^i) + \mathbb{E}(x_j^0) \\
&\geq \frac{1}{k} \sum_{i=1}^{n-t} w(u_i) + \frac{1}{k-1} \sum_{i=1}^{n-t} e_i + x_j^0 \\
&\geq \frac{1}{k} \left(w_1 - \sum_{i=1}^{t} w(v_i) \right) + \frac{1}{k-1} \left(w_2 - \sum_{e \subseteq V_1} w(e) \right) \\
&\geq \frac{1}{k} w_1 + \frac{1}{k-1} w_2 - \left(\frac{1}{k} \sum_{i=1}^{t} w(v_i) + \frac{1}{k-1} \sum_{e \subseteq V_1} w(e) \right) \\
&\geq \frac{1}{k} w_1 + \frac{1}{k-1} w_2 - \lambda \left(\frac{1}{k} t + \frac{1}{k-1} e(V_1) \right).
\end{aligned}$$

Now changing the color of u_i only affects x_j^{n-t} by at most $d(u_i)\lambda + w(u_i) \leq (d(u_i)+1)\lambda$. Hence, as in the proof of Theorem 2.3 we apply Lemma 2.1 to conclude that for $j = 1, \ldots, k$,

$$\mathbb{P}\left(x_j^{n-t}\right) < \mathbb{E}(x_j^{n-t}) - z \leq \exp\left(-\frac{z^2}{24\lambda^2 m^{2-\alpha}} \right).$$

Pick $z = \sqrt{24 \ln k} m^{1-\frac{\alpha}{2}}$; then

$$\mathbb{P}\left(x_j^{n-t} < \mathbb{E}(x_j^{n-t}) - z\right) < \exp\left(-\ln k\right) = \frac{1}{k}.$$

So there exists a partition $V(G) = X_1 \cup X_2 \cup \cdots \cup X_k$ such that for $j = 1, \ldots, k$,

$$\begin{aligned}
\tau(X_j) &\geq \mathbb{E}(x_j^{n-t}) - z \\
&\geq \frac{1}{k} w_1 + \frac{1}{k-1} w_2 - \lambda \left(\frac{1}{k} t + \frac{1}{k-1} e(V_1)\right) - z \\
&\geq \frac{1}{k} w_1 + \frac{1}{k-1} w_2 + \lambda \cdot o(m),
\end{aligned}$$

where the $o(m)$ term in the expression is

$$-\left(\frac{1}{k} m^\alpha + \frac{1}{2(k-1)} m^{2\alpha} + \sqrt{24 \ln k} m^{1-\frac{\alpha}{2}}\right).$$

Picking $\alpha = \frac{2}{5}$ to minimize $\max\{2\alpha, 1 - \alpha/2\}$, the $o(m)$ term becomes $O(m^{\frac{4}{5}})$. ∎

Suppose G is a hypergraph whose edges have size 1 or 2. We may view G as a weighted graph with weight function w such that $w(e) = 1$ for all $e \in E(G)$ with $|e| = 2$, $w(v) = 1$ for all $v \in V(G)$ with $\{v\} \in E(G)$, and $w(v) = 0$ for all $v \in V(G)$ with $\{v\} \notin E(G)$. Theorem 3.4 then gives the following result.

Theorem 3.5 *Let $k \geq 3$ be an integer and let G be a hypergraph with m_i edges of size i, $i = 1, 2$. Then there is a partition $V_1, \ldots, V_k$ of $V(G)$ such that for $i = 1, \ldots, k$,*

$$d(V_i) \geq \frac{m_1}{k} + \frac{m_2}{k-1} + O(m_2^{4/5}).$$

Note that if $X_1, \ldots, X_k$ is a random k-partition in a hypergraph with m_i edges of size i for $i = 1, 2$, then $\mathbb{E}(d(X_i)) = m_1/k + (2k-1)m_2/k^2$.

We have the following corollary, which establishes a conjecture of Bollobás and Scott [7] for large graphs.

Corollary 3.6 *Let G be a graph with m edges and let $k \geq 3$ be an integer. Then there is an integer $f(k)$ such that if $m \geq f(k)$ then $V(G)$ has a partition $V_1, \ldots, V_k$ such that $d(V_i) \geq 2m/(2k-1)$ for $i = 1, \ldots, k$.*

Note that our proof of Theorem 3.4 gives $f(k) = O(k^{10} (\log k)^{5/2})$.

4 k-Partitions – bounding edges inside each set

Bollobás and Scott [4] proved that every graph with m edges can be partitioned into k sets each of which contains at most $m/\binom{k+1}{2}$ edges, with K_{k+1} as the unique extremal graph. For large graphs, they prove in [6] that this bound can be improved to $(1 + o(1))m/k^2$.

Bollobás and Scott conjecture in [8] that any hypergraph with m_i edges of size i, $i = 1, 2$, admits a partition into k sets each of which contains at most $m_1/k + m_2/\binom{k+1}{2} + O(1)$ edges. We now prove this conjecture, using a similar approach as before. The following two lemmas will enable us to choose appropriate probabilities in a random process.

Lemma 4.1 *Let $\delta \geq 0$ and, for integers $k \geq l \geq 1$, let $a_i = a > 0$ for $i = 1, \ldots, l$ and $a_j = 0$ for $j = l+1, \ldots, k$. Suppose $\delta + a_i > 0$ for all $1 \leq i \leq k$. Then*

$$\frac{1}{\sum_{i=1}^{k} \frac{1}{\delta + a_i}} \leq \frac{\delta}{k} + \frac{1}{k^2} \sum_{i=1}^{k} a_i.$$

Proof. If $l = k$ then the inequality holds with equality (both sides equal to $(\delta + a)/k$). So we may assume $k > l$. Then $\delta > 0$, since $\delta + a_k > 0$ by assumption. Thus $\sum_{i=1}^{k} \frac{1}{\delta + a_i} = \frac{l}{\delta + a} + \frac{k-l}{\delta}$ and $\sum_{i=1}^{k} a_i = la$. Hence

$$\frac{1}{\sum_{i=1}^{k} \frac{1}{\delta + a_i}} - \left(\frac{\delta}{k} + \frac{1}{k^2} \sum_{i=1}^{k} a_i \right) = \frac{-l(k-l)a^2}{k^2(k\delta + (k-l)a)} \leq 0.$$

Thus the assertion of the lemma holds. ∎

Lemma 4.2 *Let $\delta \geq 0$ and let $a_i \geq 0$ for $i = 1, \ldots, k$. Then there exist $p_i \in [0, 1]$, $i = 1, \ldots, k$, such that $\sum_{i=1}^{k} p_i = 1$ and, for $1 \leq i \leq k$,*

$$(\delta + a_i) p_i \leq \frac{\delta}{k} + \frac{1}{k^2} \sum_{i=1}^{k} a_i.$$

Proof. If there exists some $1 \leq i \leq k$ such that $\delta + a_i = 0$, then $\delta = a_i = 0$. In this case let $p_i = 1$ and $p_j = 0$ for $j \neq i, 1 \leq j \leq k$. Then $(\delta + a_i)p_i = 0$ for $i = 1, \ldots, k$; and clearly the assertion of the lemma holds.

Therefore, we may assume that $\delta + a_i > 0$, $1 \leq i \leq k$. Setting $(\delta + a_i)p_i = (\delta + a_1)p_1$ for $i = 2, \ldots, k$, we have $p_i = \frac{\delta + a_1}{\delta + a_i} p_1$. Requiring $\sum_{i=1}^{k} p_i = 1$ we have

$$(\delta + a_1) p_1 \sum_{i=1}^{k} \frac{1}{\delta + a_i} = 1.$$

Hence for $i = 1, \ldots, k$,

$$(\delta + a_i) p_i = \frac{1}{\sum\limits_{i=1}^{k} \frac{1}{\delta + a_i}}.$$

Let

$$f(a_1, a_2, \ldots, a_k) := \frac{1}{\sum\limits_{i=1}^{k} \frac{1}{\delta + a_i}} - \left(\frac{\delta}{k} + \frac{1}{k^2} \sum_{i=1}^{k} a_i \right).$$

We need to show $f \leq 0$. This is clear if $a_i = 0$ for $i = 1, \ldots, k$, since $f(0, \ldots, 0) = 0$. Set $\alpha = \sum_{j=1}^{k} a_j$.

Let $g_l(a_1, \ldots, a_l) := f(a_1, \ldots, a_l, 0, \ldots, 0)$ for $l = 1, \ldots, k$. We now show that $g_l \leq 0$ on $D_l := [0, \alpha]^l$ for all $1 \leq l \leq k$; and hence $f = g_k \leq 0$. We apply induction on l.

Suppose $l = 1$. Clearly, $g_1(0) = f(0, 0, \ldots, 0) = 0$; and if $a_1 = a > 0$ then by Lemma 4.1, $g_1(a_1) = f(a_1, 0, \ldots, 0) \le 0$.

Therefore, we may assume $l \ge 2$. It suffices to prove $g_l(a_1, \ldots, a_l) \le 0$ for all points $(a_1, \ldots, a_l)$ that are on the boundary of D_l or critical points of g_l in D_l.

Let $(a_1, \ldots, a_l)$ be a point on the boundary of D_l. Then there exists $j \in \{1, \ldots, l\}$ such that $a_j = 0$ or $a_j = \alpha$. Since g_l is a symmetric function, we may assume $a_l = 0$ or $a_1 = \alpha$. If $a_l = 0$ then $g_l(a_1, \ldots, a_{l-1}, 0) = g_{l-1}(a_1, \ldots, a_{l-1}) \le 0$, by induction hypothesis. If $a_1 = \alpha$ then $a_2 = \ldots = a_k = 0$, and so $g_l(a_1, \ldots, a_l) = g_1(a_1) \le 0$ by induction basis.

Hence it remains to prove $g_l \le 0$ at its critical points in D_l, subject to $\sum_{j=1}^{l} a_j - \alpha = 0$. Note that for all $j = 1, \ldots, l$,

$$\frac{\partial f}{\partial a_j} = \frac{1}{\left(\sum_{i=1}^{k} \frac{1}{\delta + a_i}\right)^2} \cdot \frac{1}{(\delta + a_j)^2} - \frac{1}{k^2}.$$

Note that $\frac{\partial g_l}{\partial a_j}$ is obtained from $\frac{\partial f}{\partial a_j}$ by setting $a_{l+1} = \ldots = a_k = 0$. Thus, letting $\dfrac{\partial g_l}{\partial a_j} = \lambda$ (the Lagrange multiplier) for $j = 1, \ldots, l$, we have for $1 \le s \ne t \le l$,

$$\frac{1}{\left(\sum_{i=1}^{k} \frac{1}{\delta + a_i}\right)^2} \cdot \frac{1}{(\delta + a_s)^2} - \frac{1}{k^2} = \frac{1}{\left(\sum_{i=1}^{k} \frac{1}{\delta + a_i}\right)^2} \cdot \frac{1}{(\delta + a_t)^2} - \frac{1}{k^2}.$$

As a consequence, $(\delta + a_s)^2 = (\delta + a_t)^2$ for $1 \le s \ne t \le l$, which implies $a_s = a_t$. Thus, if $(a_1, a_2, \ldots, a_l)$ is a critical point of g_l in D_l, then there exists $a > 0$ such that $a_i = a > 0$ for $i = 1, \ldots, l$. So $g_l \le 0$ by Lemma 4.1. ∎

We now prove the following partition result for weighted graphs.

Theorem 4.3 *Let G be a graph with m edges, and let $w : V(G) \cup E(G) \to \mathbf{R}^+$ such that $w(e) > 0$ for all $e \in E(G)$. Let $\lambda := \max\{w(x) : x \in V(G) \cup E(G)\}$, $w_1 = \sum_{v \in V(G)} w(v)$ and $w_2 = \sum_{e \in E(G)} w(e)$. Then for any integer $k \ge 1$ there is a partition $X_1, \ldots, X_k$ of $V(G)$ such that for $i = 1, \ldots, k$,*

$$e(X_i) \le \frac{1}{k} w_1 + \frac{1}{k^2} w_2 + \lambda \cdot O(m^{4/5}).$$

Proof. We may assume that G is connected. We use the same notation as in the proof of Theorem 2.3. Let $V(G) = \{v_1, \ldots, v_n\}$ such that $d(v_1) \ge d(v_2) \ge \ldots \ge d(v_n)$. Let $V_1 = \{v_1, \ldots, v_t\}$ with $t = \lfloor m^\alpha \rfloor$, where $0 < \alpha < 1/2$; and let $V_2 := V(G) \setminus V_1 = \{u_1, \ldots, u_{n-t}\}$ such that $e(u_i, V_1 \cup \{u_1, \ldots, u_{i-1}\}) > 0$ for $i = 1, \ldots, n - t$. Then $e(V_1) \le \frac{1}{2} m^{2\alpha}$ and $d(v_{t+1}) \le 2m^{1-\alpha}$.

Fix a random k-partition $V_1 = Y_1 \cup Y_2 \cup \cdots \cup Y_k$, and assign each member of Y_i the color i, $1 \le i \le k$. Extend this coloring to $V(G)$, where each vertex $u_i \in V_2$ is independently assigned the color j with probability p_j^i, where $\sum_{j=1}^{k} p_j^i = 1$. Let Z_i denote the indicator random variable of the event of coloring u_i. Hence $Z_i = j$ iff u_i is assigned the color j.

Let $G_i = G[V_1 \cup \{u_1, \cdots, u_i\}]$ for $i = 1, \ldots, n - t$, and let $G_0 = G[V_1]$. For $j = 1, \ldots, k$, let

$X_j^0 = Y_j$ and $x_j^0 = w(X_j^0)$; and for $i = 1, \dots, n-t$ and $j = 1, \dots, k$, define

$$\begin{aligned} &X_j^i = \{\text{vertices of } G_i \text{ with color } j\}, \\ &x_j^i = w(X_j^i), \\ &\Delta x_j^i = x_j^i - x_j^{i-1}, \\ &a_j^i = \sum_{e \in (u_i, X_j^{i-1})} w(e). \end{aligned}$$

Note that a_j^i is determined by $(Z_1, \dots, Z_{i-1})$. Hence for $1 \le i \le n-t$ and $1 \le j \le k$,

$$\mathbb{E}(\Delta x_j^i | Z_1, \dots, Z_{i-1}) = (w(u_i) + a_j^i) p_j^i,$$

and so

$$\mathbb{E}(\Delta x_j^i) = (w(u_i) + b_j^i) p_j^i,$$

where here

$$b_j^i = \sum_{(Z_1, \dots, Z_{i-1})} \mathbb{P}(Z_1, \dots, Z_{i-1}) a_j^i.$$

Since a_j^i is determined by $(Z_1, \dots, Z_{i-1})$, b_j^i is determined by p_j^s, $1 \le j \le k$ and $1 \le s \le i-1$. Note that $e_i := \sum_{j=1}^{k} a_j^i = \sum_{e \in (u_i, G_{i-1})} w(e) > 0$, which is independent of $Z_1, \dots, Z_{n-t}$. By Lemma 4.2, there exist $p_j^i \in [0,1]$, $1 \le j \le k$, such that $\sum_{j=1}^k p_j^i = 1$ and, for $1 \le i \le n-t$ and $j = 1, \dots, k$,

$$\begin{aligned} \mathbb{E}(\Delta x_j^i) &\le \frac{w(u_i)}{k} + \frac{1}{k^2} \sum_{j=1}^{k} b_j^i \\ &= \frac{w(u_i)}{k} + \frac{1}{k^2} \sum_{j=1}^{k} \sum_{(Z_1, \dots, Z_{i-1})} \mathbb{P}(Z_1, \dots, Z_{i-1}) a_j^i \\ &= \frac{w(u_i)}{k} + \frac{1}{k^2} \sum_{(Z_1, \dots, Z_{i-1})} \left(\mathbb{P}(Z_1, \dots, Z_{i-1}) \sum_{j=1}^{k} a_j^i \right) \\ &= \frac{w(u_i)}{k} + \frac{1}{k^2} \sum_{(Z_1, \dots, Z_{i-1})} \mathbb{P}(Z_1, \dots, Z_{i-1}) e_i \\ &= \frac{w(u_i)}{k} + \frac{1}{k^2} e_i. \end{aligned}$$

Note that each p_j^i is determined by b_l^i, $1 \le l \le k$; and hence each p_j^i is recursively defined by

p_l^s, $1 \le l \le k$ and $1 \le s \le i-1$. Also note that $w_2 = \sum_{e \in E(G_0)} w(e) + \sum_{i=1}^{n-t} e_i$. Now

$$\begin{aligned}
\mathbb{E}(x_j^{n-t}) &= \sum_{i=1}^{n-t} \mathbb{E}(\Delta x_j^i) + \mathbb{E}(x_j^0) \\
&\le \frac{1}{k} \sum_{i=1}^{n-t} w(u_i) + \frac{1}{k^2} \sum_{i=1}^{n-t} e_i + x_j^0 \\
&\le \frac{1}{k} w_1 + \frac{1}{k^2} w_2 + \sum_{i=1}^{t} w(v_i) + \sum_{e \subseteq V_1} w(e) \\
&\le \frac{1}{k} w_1 + \frac{1}{k^2} w_2 + \lambda(t + e(V_1)).
\end{aligned}$$

Clearly, changing the color of u_i affects x_j^{n-t} by at most $d(u_i)\lambda + w(u_i) \le (d(u_i)+1)\lambda$. As in the proof of Theorem 2.3, we apply Lemma 2.1 to conclude that

$$\mathbb{P}\left(x_j^{n-t} > \mathbb{E}(x_j^{n-t}) + z\right) \le \exp\left(-\frac{z^2}{2\lambda^2 \sum_{i=1}^{n-t}(d(u_i)+1)^2}\right) \le \exp\left(-\frac{z^2}{24\lambda^2 m^{2-\alpha}}\right).$$

Let $z = \lambda\sqrt{24 \ln k}\, m^{1-\frac{\alpha}{2}}$. Then

$$\mathbb{P}\left(x_j^{n-t} > \mathbb{E}(x_j^{n-t}) + z\right) < \exp(-\ln k) = \frac{1}{k}.$$

So there exists a partition $V(G) = X_1 \cup X_2 \cup \cdots \cup X_k$, such that for $1 \le j \le k$,

$$\begin{aligned}
e(X_j) &\le \mathbb{E}(x_j^{n-t}) + z \\
&\le \frac{1}{k} w_1 + \frac{1}{k^2} w_2 + \lambda\left(t + e(V_1)\right) + z \\
&\le \frac{1}{k} w_1 + \frac{1}{k^2} w_2 + \lambda \cdot o(m).
\end{aligned}$$

The $o(m)$ term in the expression is

$$m^\alpha + \frac{1}{2} m^{2\alpha} + \sqrt{24 \ln k}\, m^{1-\frac{\alpha}{2}}.$$

Picking $\alpha = \frac{2}{5}$ to minimize $\max\{2\alpha, 1-\alpha/2\}$, the $o(m)$ term becomes $O(m^{\frac{4}{5}})$. ∎

For a hypergraph G with edges of size 1 or 2, we may view G as a weighted graph with weight function w such that $w(e) = 1$ for all $e \in E(G)$ with $|e| = 2$, $w(v) = 1$ for all $v \in V(G)$ with $\{v\} \in E(G)$, and $w(v) = 0$ for $v \in V(G)$ with $\{v\} \notin E(G)$. Then Theorem 4.3 gives the following result, establishing a conjecture of Bollobás and Scott [8] (the case $m_1 = o(m_2)$ is implied by equation (2) in [8]).

Theorem 4.4 *Let G be a hypergraph with m_i edges of size i, $i = 1, 2$. Then for any integer $k \ge 1$, there is a partition $X_1, \ldots, X_k$ of $V(G)$ such that for $i = 1, \ldots, k$,*

$$e(X_i) \le \frac{m_1}{k} + \frac{m_2}{k^2} + O(m_2^{4/5}).$$

Note that the term $m_1/k + m_2/k^2$ is the expected value of $e(X_i)$ if $X_1, \ldots, X_k$ is a random partition. Bollobás and Scott further ask in [8] whether $O(m_2^{4/5})$ in Theorem 4.4 can be imporved to $O(\sqrt{m_1 + m_2})$.

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